

# **A classical two-body Hamiltonian model and its mean field approximation**

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## **Abstract**

We extend a recent billiard model of the nuclear  $N$ -body Hamiltonian to consider a finite two-body interaction. This permits a treatment of the Hamiltonian by a mean field theory, and also allows the possibility to model reactions between nuclei. The density and the mean field potential can be accurately described by a scaling function which shows the qualitative features of the liquid drop picture of the nucleus.

The nuclear many-body problem is often approximated by phenomenological mean field or liquid drop models which serve as good starting points to study the shape and the spectrum of low lying single particle and collective excitations [1–3]. This well-known approach is a consequence of the inadequate knowledge of the microscopic nuclear Hamiltonian, and the difficulty in dealing with the many-body problem. The mean field approximation is commonly employed to make the many-body problem tractable. Therefore it is of interest to question of how well mean-field theory works for a given microscopic Hamiltonian that can be solved exactly.

A promising model for such a program may readily be obtained from a recent billiard model of the nuclear  $N$ -body Hamiltonian. This is an interacting  $N$ -body system with two-body interactions that is rich enough to show various features of self-bound many-body systems yet simply enough to allow for practical calculations and an understanding [4].

Generalizing the model of ref. [4], we consider the Hamiltonian

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + \sum_{i<j} V(|\vec{r}_i - \vec{r}_j|), \quad (1)$$

where  $\vec{r}_i$  is a three-dimensional position vector of the  $i$ -th nucleon and  $\vec{p}_i$  is its conjugate momentum and the interaction given by

$$V(r) = \begin{cases} -V_0 & \text{for } r < a, \\ 0 & \text{for } r \geq a, \end{cases} \quad (2)$$

where  $V_0$  is a positive constant. Total energy  $E$ , momentum  $\vec{P}$  and angular momentum  $\vec{L}$  are conserved quantities. Interactions alter the momenta of particles whenever they are distance  $a$  apart, and otherwise they move freely.

For  $V_0 = \infty$  one obtains a billiard model, which was recently studied in ref. [4,5]. Due to the simple form of the two-body interaction, the numerical integration of the equations of motion is quite easy, scaling with particle number as  $N \log N$  rather than the usual  $N^2$  typical of the classical  $N$ -body problem. It was found that the dynamics is dominantly chaotic and ergodic. In the center of mass system one finds a constant single-particle density

inside a circle of diameter  $a$  that quickly drops to zero within a thin surface region. The phase space structure does not depend on energy in such billiards.

Finite values of  $V_0$  introduce an energy scale and a finite binding energy of the system. The Hamiltonian then permits reactions like particle capture or emission, giving it important features for a model of the nucleus and its dynamics.

In this note we will examine the mean field theory of the  $N$ -body Hamiltonian (1,2). To this purpose we introduce the single-particle phase space density  $f(\vec{r}, \vec{p})$ . Let  $p^2/2m + W(\vec{r})$  be the mean field Hamiltonian. As is well known, any density which is a functional of the mean field Hamiltonian solves the corresponding Vlasov equation [6]. However, we are interested in the situation corresponding to thermal equilibrium and choose the canonical distribution function

$$f(\vec{r}, \vec{p}) \propto \exp\left(-\frac{p^2/2m + W(\vec{r})}{kT}\right), \quad (3)$$

where  $T$  denotes the temperature. Obviously, the phase space distribution function is a product of the normalized momentum space density  $[4(2\pi mkT)^{3/2}]^{-1} \exp(-p^2/2mkT)$  and the number density

$$n(\vec{r}) = c \exp\left(-\frac{W(\vec{r})}{kT}\right). \quad (4)$$

The normalization  $c$  is given by

$$c = N \left[ \int d^3r n(\vec{r}) \right]^{-1}$$

The mean field potential  $W(\vec{r})$  depends on the number density and the two-body interaction

$$W(\vec{r}) = \int d^3x V(|\vec{r} - \vec{x}|) n(\vec{x}). \quad (5)$$

Eq.(4) together with eq.(5) are the mean field equations to be solved self-consistently.

We next observe that the solution of the mean field equations (4-5) will have an interesting scaling property. Let  $W(\vec{r})$  and  $n(\vec{r})$  denote the mean field potential and density for parameters  $N, V_0$  and  $kT$ . Then  $\alpha W(\vec{r})$  is the mean field potential for parameters  $\alpha N V_0$

and  $\alpha kT$ , while  $\alpha n(\vec{r})$  is the density for parameters  $\alpha N$  and  $V_0/\alpha kT$ . Thus, the normalized quantities  $V(\vec{r})/NV_0$  and  $n(\vec{r})/N$  depend only on the ratio

$$\gamma \equiv NV_0/kT \quad (6)$$

of parameters. This scaling behavior comes directly from eqs. (4,5) and does not depend on the specific form of the two-body interaction (2).

It is well known that the density of an open system can be represented as a thermal distribution function only for  $\gamma \gg 1$ . In this case the density is exponentially small outside the nucleus and can be omitted. In what follows we restrict ourselves to this low temperature regime.

We are interested in spherical symmetric solutions of the mean field equations. In this case eq.(5) may be written as

$$W(r) = \int_0^\infty dx n(x) v(r, x), \quad (7)$$

where

$$v(r, x) = -2\pi V_0 \begin{cases} 0 & \text{for } a < |r - x|, \\ 2x^2 & \text{for } a > r + x, \\ \frac{x}{2r} [a^2 - (r - x)^2] & \text{for } |r - x| < a < r + x. \end{cases} \quad (8)$$

For computational purposes it is useful to rewrite the mean field potential as

$$\begin{aligned} W(r) = -2\pi V_0 \Big[ & \Theta(a - r) \int_0^{a-r} dx 2x^2 n(x) \\ & + \Theta(a - r) \int_{\max(a-r, r)}^{a+r} dx \frac{x}{2r} (a^2 - (x - r)^2) n(x) \\ & + \Theta(r - a/2) \int_{|r-a|}^r dx \frac{x}{2r} (a^2 - (x - r)^2) n(x) \Big] \end{aligned} \quad (9)$$

where  $\Theta(x)$  is the unit step function.

We solve the mean field equations numerically by iteration. We take an initial density  $n(r)$  that is constant inside a sphere of diameter  $a$  and zero outside. Upon several iterations

one obtains a converged solution. Fig. 1 shows the resulting number density  $n(r)/N$  and mean field potential  $W(r)/NV_0$  for different values of the parameter  $\gamma$ . For large  $\gamma$  the mean field potential is flat for  $r < a/2$  and increases until  $r \approx 3a/2$  while the density is constant for  $r < a$  and quickly drops to zero around  $r \approx a/2$ . Decreasing values of  $\gamma$  (i.e. decreasing values of  $V_0$  and/or increasing temperatures  $T$  while the number of particles  $N$  is kept fixed) lead to a thicker surface region and to a more extended mean field potential. However, for  $r < a$  the potential is still a flat bottom potential and the density is roughly constant. Thus, the system exhibits qualitative features of the liquid drop model. However, our simple model does not yield a saturation of the density or the binding energy with increasing  $N$ . This is due to the absence of a repulsive potential core.

In the limit  $\gamma \rightarrow \infty$  the density approaches that of a sharp-edged liquid drop, constant inside a sphere of radius  $a/2$  and zero outside. For finite values of  $\gamma$ , the surface thickness can be estimated assuming a trapezoidal density distribution. Taking the width parameter as  $\epsilon$  and demanding self-consistency in the neighborhood of  $r = a/2$ , one finds the relation

$$\epsilon/a = (3\gamma)^{-1/2}. \quad (10)$$

It is also interesting to compare the mean field results with the interacting system (1,2). Agreement may only be expected for sufficiently large numbers of particles  $N_0$ . In what follows we set  $N_0 = 300$ . The initial conditions of the  $N_0$ -body system are chosen such that the positions are drawn at random inside a sphere of diameter  $a$  while the momenta are drawn from a Maxwell-Boltzmann distribution with temperature  $T$ . We follow the time evolution of the many-body system for roughly  $10^5$  interactions to allow for equilibration. At the end of the evolution we take the positions in the center of mass system and compute the integrated density

$$N(r) = 4\pi \int_0^r dx x^2 n(x) \quad (11)$$

by counting the number of particles inside a sphere of diameter  $r$ . Fig. 2 compares the average of ten runs with the mean field result for various values of the temperature. The agreement is rather good confirming the validity of the mean field approximation.

Finally, we ask what values the parameters should have to correspond to physical properties of nuclei. Let us consider a nucleus of mass  $A \approx 75$ . Then the radius is given by  $a/2 \approx 1.2A^{1/3} \approx 5$  fm. The liquid-drop surface thickness is roughly  $\epsilon \approx 1$  fm, yielding from eq. (10)  $\gamma \approx 30$ . We would like to set the energy scale by the depth of the mean field potential, which should be  $NV_0 \approx 50$  MeV to correspond to a typical Woods-Saxon potential of nuclear physics. Turning to eq. (7), we see that the two parameter values can be obtained taking the temperature as  $kT \approx 2$  MeV. Equilibrated nuclei at such a temperature can be easily produced in heavy ion reactions, so the model might have some applicability to nuclear reactions. However, in detail the Fermionic nature of the nuclear many-body problem will make considerable differences from the present classical model. In particular, the chemical potentials at a given temperature are very different, which would be important for the nucleon evaporation rates.

## ACKNOWLEDGMENTS

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# FIGURES

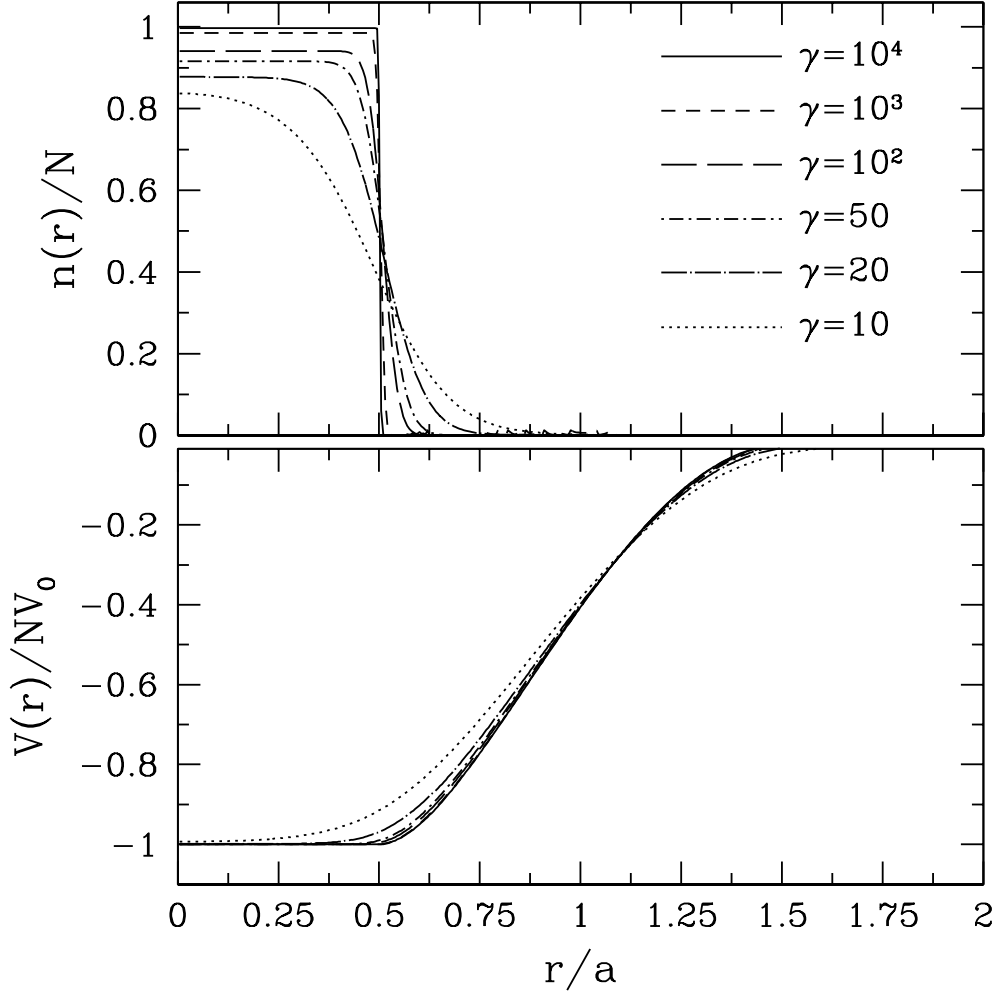


FIG. 1. Mean field potentials (lower panels) and densities (upper panels) for various values of the parameter  $\gamma = NV_0/kT$ .



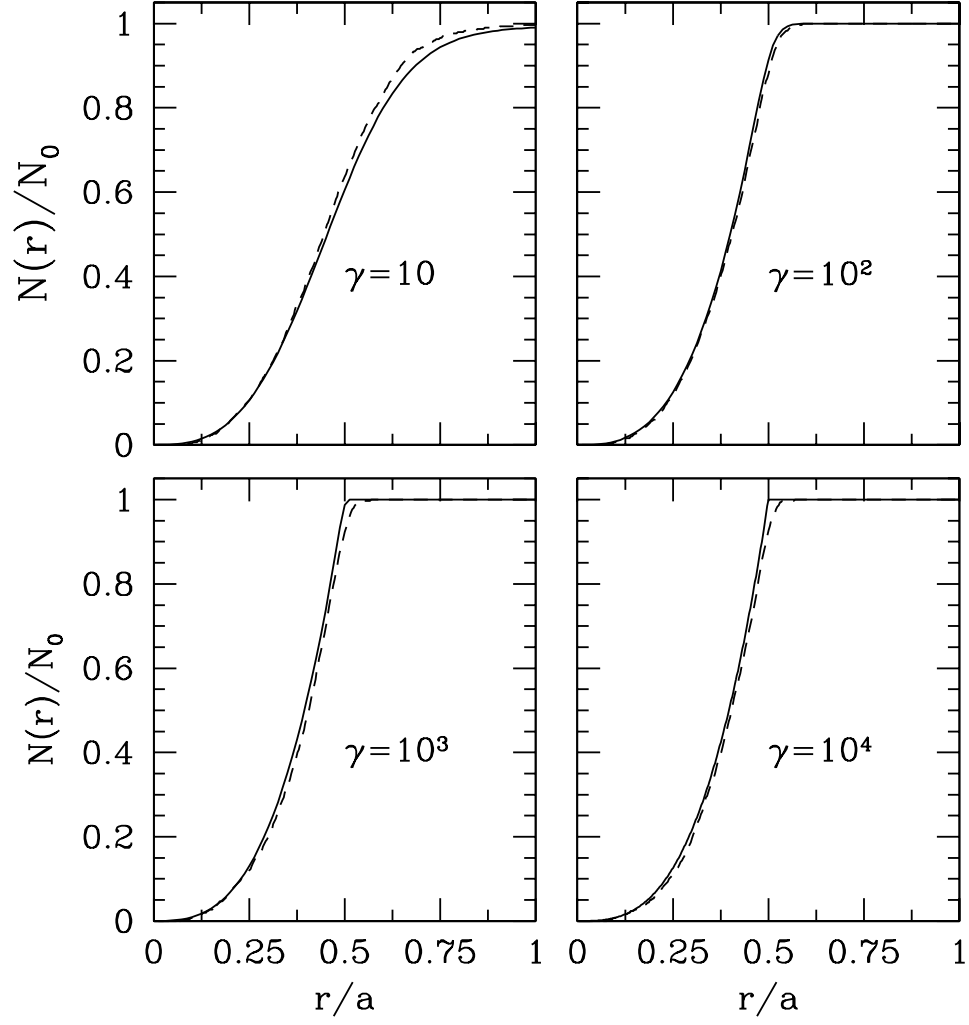


FIG. 2. Integrated number density from microscopic calculations (dashed lines) compared to mean field results (solid lines) for different values of the parameter  $\gamma = NV_0/kT$